Plan: (1) time contours, (2) Green functions, (3) Wick's theorem

Last time: $S(t_1, t_2) = T \exp\left[-i\int_{t_2}^{t_1} dt' \, \hat{V}(t')\right]$

(1) Gell-Man Low formalism:

- So far we have a formalism to evolve wave functions in time
- We still need the initial wave functions!

Solution:

- let ϕ_0 be the ground state of H_0
- Let us set $\hat{\Psi}(-\infty) = \phi_0$ [that is in the "dim past"]
- Turn on the interactions slowly

$\Rightarrow$ $\hat{\Psi}(t)$ evolves adiabatically

$\Rightarrow$ $\hat{\Psi}(t)$ becomes the ground state of $H = H_0 + V$ in the present

$\Rightarrow$ $\hat{\Psi}(t) \equiv S(t, -\infty)\, \hat{\Psi}(-\infty)$

- To compute operators $\langle \hat{O}(t) \rangle$, we also need the "bra"

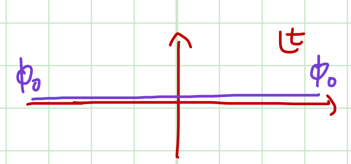
$\Rightarrow$ "cheat" turn off the interactions slowly so that in the dim future $\hat{\Psi}(\infty) \to \phi_0$

$$\Rightarrow \langle \underbrace{\hat{\Psi}^+(\infty)}_{\phi_0} S(\infty, t)\, \hat{O}(t)\, S(t, -\infty) \underbrace{\hat{\Psi}(-\infty)}_{\phi_0} \rangle \quad \text{[Gell-Mann Low]}$$

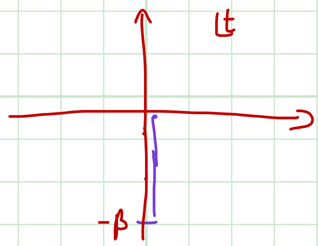
note 1: $1 = \langle \phi_0 | \phi_0 \rangle = \underbrace{e^{iL}}_{\text{phase factor}} \langle S(\infty, -\infty) \rangle$ $\Rightarrow$ $\langle \hat{O}(t) \rangle = \dfrac{\langle S(\infty, t)\, \hat{O}(t)\, S(t, -\infty) \rangle}{\langle S(+\infty, -\infty) \rangle}$

Note 2: there are alternative formalisms where "cheating" is not required. We will return to this topic once we are comfortable with diagrams

Gell-Man Low contour



Matsubara

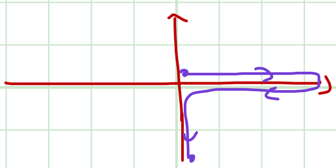


Motivation:

$\Rightarrow$ thermodynamic Trace $\Rightarrow e^{-\beta H}$

$\Rightarrow$ take e^{-iHt}, $\quad t \to -i\beta$

Keldysh



(1) No need to go to infinite future, return to past $\Rightarrow$ so no phase

(2) $\Rightarrow$ If I want time evolve a thermodynamic state, combine with In time evolution.

(2) Green('s) functions.

- Electron Green's function

$\Rightarrow$ Interacting System

$\Rightarrow$ Prototype for calculating correlation functions (e.g. Susceptibilities)

$$G(k,t,t') = G(k, t-t') = -i \langle T(\underbrace{c_k(t)}_{e^{iHt} c_k e^{-iHt}} \; c_k^{+}(t')) \rangle \longleftarrow \text{def. in Heisenberg picture}$$

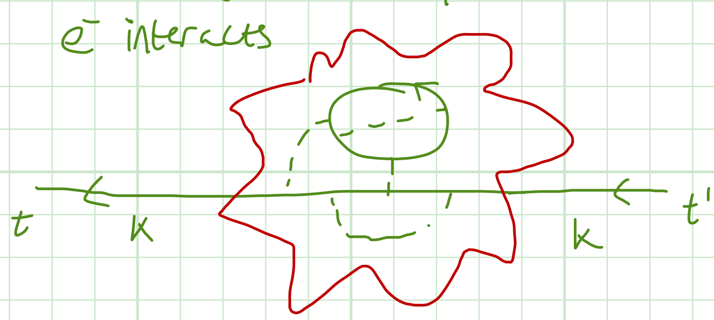
What does the Green's function mean?

For $t > t'$: $\quad G(k, t>t') = -i \langle c_k(t) \; c_k^{+}(t') \rangle$

remove e^- at t $\qquad$ inject e^- at time t'

k not eigenstate of H

e^- interacts



For $t < t'$: $\quad G(k, t<t') = i \langle c_k^{+}(t') \, c_k(t) \rangle$

note sign due to anti-commutation relation

$\Rightarrow$ remove electron first + put it back later

computing the Green's function using interaction picture:

$$c_k(t) = e^{iHt} e^{-iH_0t} \underbrace{e^{iH_0t} c_k e^{-iH_0t}}_{\hat{c}_k(t)} e^{iH_0t} e^{-iHt} = U^+(t)\, \hat{c}_k(t)\, U(t)$$

[interaction picture]

$$= S(0,t)\,\hat{c}_k(t)\,S(t,0)$$

Hence: $G(k,t-t') = -i\,\theta(t-t')\,\langle c_k(t)\, c_k^+(t')\rangle + i\,\theta(t'-t)\,\langle c_k^+(t')\, c_k(t)\rangle$

$$= -i\,\theta(t-t')\,\langle S(-\infty,t)\,\hat{c}_k(t)\,S(t,t')\,\hat{c}_k^+(t')\,S(t',-\infty)\rangle$$

$$+ i\,\theta(t'-t)\,\langle S(-\infty,t')\,\hat{c}_k(t')\,S(t',t)\,\hat{c}_k(t)\,S(t,-\infty)\rangle$$

Note: $\langle S(-\infty,t)| = \dfrac{\langle S(+\infty,-\infty)\, S(\infty,t)|}{\langle S(+\infty,-\infty)\rangle}$ this part, we argued just adds a phase

Using this fact, the Green's function becomes:

$$G(k,t-t') = -i\,\theta(t-t')\,\frac{\langle S(\infty,t)\,\hat{c}_k(t)\,S(t,t')\,\hat{c}_k^+(t')\,S(t',-\infty)\rangle}{\langle S(\infty,-\infty)\rangle} + i\theta(t'-t)\,[\ldots]$$

$$= -i\,\frac{\langle T[\hat{c}_k(t)\,\hat{c}_k^+(t')\,S(\infty,-\infty)]\rangle}{\langle S(\infty,-\infty)\rangle}$$

$\Rightarrow$ T operator makes sure that all parts of Green's function appear in the correct chronological order

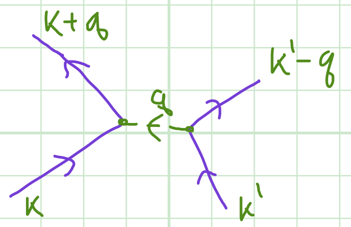
$\Rightarrow$ QFT people $\Rightarrow$ why the denomenator?

$\Rightarrow$ Now we are ready to plug in the Dyson series for the S-matrix

$$G(k,t-t') = \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n \frac{\langle T\ \hat{c}_k(t)\,\hat{V}(t_1)\ldots\hat{V}(t_n)\,\hat{c}_k^+(t')\rangle}{\langle S(\infty,-\infty)\rangle}$$

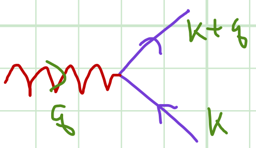
Interactions: What type of interactions are important in CM?

(1) $e^- - e^-$ $\Rightarrow$ This is really mediated via a virtual photon. Most of the time we can



ignore that fact:

$$V = \frac{e^2}{4\pi\epsilon}\int \frac{n(r)\,n(r')}{|r-r'|}\,dr\,dr' = \frac{1}{2}\sum_{kk'q} \frac{4\pi e^2}{q^2}\, c^{+}_{k+q}\, c_k\, c^{+}_{k'-q}\, c_{k'}$$



(2) $\bar{e} - ph \Rightarrow$ phonon is associated with the distortion of the atomic lattice $\Rightarrow$ positively charged $\Rightarrow$ this distortion can scatter a negatively charged $\bar{e}$.

$$V_{int} = \sum_{k,q} i M_{k,q}\, c^{+}_{k+q}\, c_k \left[b_q - b^{+}_{-q} \right]$$

To make progress we must compute expectation values of time ordered strings of operators.

e.g. $\langle T \hat{c}_k(t)\, \underbrace{\hat{c}_p(t_1)\, \hat{c}^{+}_q(t_2)}_{\text{these may be associated with scattering a phonon}}\, \hat{c}^{+}_k(t') \rangle$

(3) Wick's theorem: Enforce time ordering by splitting the string into all pairs + time ordering the pairs

$$= -\langle T\, \hat{c}_k(t)\, \hat{c}^{+}_q(t_2)\rangle \langle T \hat{c}_p(t_1)\, \hat{c}^{+}_k(t')\rangle + \langle T \hat{c}_k(t)\, \hat{c}^{+}_k(t')\rangle \langle T\, \hat{c}_p(t_1)\, \hat{c}^{+}_q(t_2)\rangle$$

↑ not the sign change induced by $\{c_p(t_1), c^{+}_q(t_2)\}$

(1) D.S. book has an equivalent def. in terms of contractions + normal-ordered products $\Rightarrow$ we will practice these def. by end of lecture + in home work

(2) Whenever we interchange two fermion operators $\Rightarrow$ add "−"

(3) for strings of operators of different types, e.g. $\bar{e}$ + ph we can split the $\bar{e}$ + ph expectation values as the operators commute

$$\langle T\, \hat{c}^{+}_{p+q}(t_1)\, \hat{b}_q(t_1)\, \hat{c}_p(t_1)\, \hat{c}^{+}_{p'-q}(t_2)\, \hat{b}^{+}_q(t_2)\, \hat{c}_{p'}(t_2)\rangle \rightarrow \langle T\, \hat{c}^{+}_{p+q}(t_1)\, \hat{c}_p(t_1)\, c^{+}_{p'-q}(t_2)\, c_{p'}(t_2)\rangle$$
$$\langle T\, \hat{b}_q(t_1)\, \hat{b}^{+}_q(t_2)\rangle$$

(4) If two operators occur at the same time, the destruction operator goes to the right

$$\langle T\, c_k^\dagger(t)\, c_{k'}(t)\rangle = \delta(k-k')\,\langle c_k^\dagger c_k\rangle = \delta(k-k')\, n_F(\xi_k)$$

(5) When two operators in a contraction have diff. time arguments we put the destruction operator on the left + recognize the result as the free Green's function

$$\langle T\, \hat{c}_k^\dagger(t_1)\, \hat{c}_{k'}(t_2)\rangle = -\delta \langle T\, \hat{c}_{k'}(t_2)\, \hat{c}_k^\dagger(t_1)\rangle = i G^{(0)}(k,k';t_1,t_2)$$
$$= i G^{(0)}(k, t_1-t_2)\, \delta_{k-k'}$$

assume translation invar. in time + space

All we have left to do is

(a) figure out what the free Green's functions are

(b) develop a method for systematically organizing the perturbation theory

(c) $\Rightarrow$ Ready to apply to problems!

Green function of the free electron:

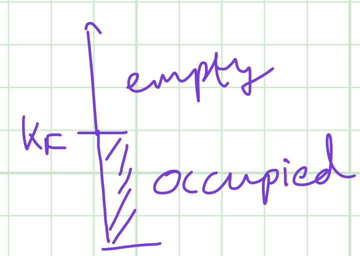
$$H_0 = \sum_k \hbar\, \omega_k\, c_k^\dagger c_k$$

$\hbar \to 1$ $\quad \omega_k = \frac{k^2}{2m} - \mu$ for electrons

Ground state:

if $k < k_F \Rightarrow$ k-th mode occupied

if $k > k_F \Rightarrow$ mode empty



$$\Rightarrow \langle c_k c_k^\dagger\rangle = \theta(k-k_F) = (1-n_F(\xi_k))$$
$$\langle c_k^\dagger c_k\rangle = \theta(k_F-k) = n_F(\xi_k)$$

Hence $G^{(0)}(k, t-t') = -i\left(\theta(t-t')\,\langle \hat{c}_k(t)\, \hat{c}_k^\dagger(t')\rangle - \theta(t'-t)\,\langle \hat{c}_k^\dagger(t')\, \hat{c}_k(t)\rangle\right)$

To complete, we need to time evolve the creation annihilation operators:

$$i\partial_t \hat{c}(t) = i\, e^{+iH_0 t}\, c\, e^{-iH_0 t} = i\left(+iH_0\, \hat{c}(t) - \hat{c}(t)\, iH_0\right) = [\hat{c}(t), H_0]$$

using the above H_0, we obtain

$$i\partial_t \hat{c}_k(t) = [e^{iH_0 t} c_k e^{-iH_0 t}, \omega_k c_k^+ c_k]$$

$$= e^{i\omega_k n_k t} c_k e^{-i\omega_k n_k t} \omega_k n_k - \omega_k \hat{n}_k e^{i\omega_k n_k t} c_k e^{-i\omega_k n_k t}$$

$$= e^{i\omega_k(n-1)t} c_k e^{-i\omega_k n_k t} \omega_k n_k - \omega(n-1) e^{i\omega(n-1)t} c_k e^{-i\omega n t}$$

$$= e^{i\omega_k(n-1)} \omega_k[n-(n-1)]\, c_k e^{-i\omega_k n t} = \omega_k \hat{c}_k(t)$$

$$i\partial_t \hat{c}_k^+(t) = -\omega_k \hat{c}_k^+(t)$$

$$\Rightarrow \quad c_k(t) = e^{-i\omega_k t} c_k$$

$$c_k^+(t) = e^{i\omega_k t} c_k^+$$

using these, we obtain: $G^{(0)}(k, t-t') = -i[\theta(t-t')\theta(\xi_k) - \theta(t'-t)\theta(-\xi_k)]\, e^{-i\omega_k(t-t')}$

Similar considerations give us the phonon Green's function see D.S.

$$G_k^{ph}(t) = -i\, e^{-i\omega_k(t)}\, \theta(t)$$

only one θ function, no occupied phonon states in equilibrium at $T=0$.

Fourier transforming:

$$G_k^{ph}(\omega) = -i\int_{-\infty}^{\infty} dt\, e^{i\omega t} e^{-i\omega_k t}\, \theta(t)$$

$$= \lim_{\epsilon\to 0^+} -i\int_0^{\infty} dt\, e^{i(\omega-\omega_k)t} e^{-\epsilon t}$$

needed for convergence

$$= \lim_{\epsilon\to 0^+} \frac{1}{\omega - \omega_k + i\epsilon}$$

$$G_k^{el}(\omega) = \frac{1}{\omega - \omega_k + i\delta_k}$$

$\delta_k = \delta\, \mathrm{sign}(\omega_k)$